

Finite difference schemes and discrete boundary layers

Benjamin BOUTIN



University Rennes 1

IRMA Strasbourg – Séminaire EDP

May 3. 2022

Context and collaborations

- **ANR Nabuco (2018-2022)** Artificial boundaries for coupling hyperbolic and dispersive models in coastal oceanography

Consistency analysis with boundary layers

with J.-F. COULOMBEL

Close to optimal semigroup estimates for the discrete hyperbolic initial boundary value problems

CEMRACS 2019 projet

with J.-F. COULOMBEL, T.H.T. NGUYEN, A. SYLLA & S. TRAN TIEN

Improvement of convergence estimates for high order numerical schemes for transport equations on bounded domains

Stability analysis

- T.H.Thương NGUYỄN, (PhD 2017-2020 adv. with N. SEGUIN)
Stiff stability analysis of discrete hyperbolic relaxation models with boundaries.
- P. LE BARBENCHON (PhD 2020-.. adv. with N. SEGUIN)
Stability analysis of high order discrete boundary conditions.

Context and collaborations

- **ANR Nabuco (2018-2022)** Artificial boundaries for coupling hyperbolic and dispersive models in coastal oceanography

Consistency analysis with boundary layers

with J.-F. COULOMBEL

Close to optimal semigroup estimates for the discrete hyperbolic initial boundary value problems

CEMRACS 2019 projet

with J.-F. COULOMBEL, T.H.T. NGUYEN, A. SYLLA & S. TRAN TIEN

Improvement of convergence estimates for high order numerical schemes for transport equations on bounded domains

Stability analysis

- T.H.Thương NGUYỄN, (PhD 2017-2020 adv. with N. SEGUIN)
Stiff stability analysis of discrete hyperbolic relaxation models with boundaries.
- P. LE BARBENCHON (PhD 2020-.. adv. with N. SEGUIN)
Stability analysis of high order discrete boundary conditions.

First experiments, problematics and strategies

Stability theory for the initial boundary value problem

Dirichlet boundary layer and semigroup estimate for discrete IBVP

High order outflowing extrapolation

Numerical experiments with inconsistent outflowing boundary condition

Leftgoing transport equation

$$\partial_t u - \partial_x u = 0, \quad x > 0, \quad t \geq 0$$

Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{1}{2} \left(\frac{\Delta t}{\Delta x} \right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n), \quad j \geq 1$$

Homogeneous Dirichlet b.c. at $x = 0$

$$u_0^n = 0$$

Test case

$$\text{initial data } u(x, 0) = e^{-100(x-0.5)^2}$$

solution computed at time $T = 0.4$ with various space steps.

Numerical experiments with inconsistent outflowing boundary condition

Another higher order scheme

Third order explicit scheme multistep (AB3 - 5pts in space)

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \left(\frac{23}{12} v_j^n - \frac{16}{12} v_j^{n-1} + \frac{5}{12} v_j^{n-2} \right)$$

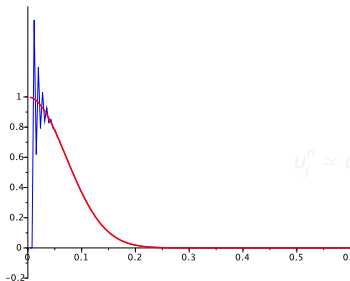
$$v_j^n := -\frac{1}{12}(-u_{j+2}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-2}^n) - \frac{1}{24}(-u_{j+2}^n + 4u_{j+1}^n - 6u_j^n + 4u_{j-1}^n - u_{j-2}^n)$$

Homogeneous Dirichlet b.c. at $x = 0$

$$u_0^n = u_1^n = 0$$

Multiscale approximation of the discrete solution

- expected accuracy partially destroyed: non-existence of discrete traces
↔ defective consistency at the boundary
- + convergence still available far from the boundary
↔ stability in some sense
- ? improve the consistency analysis and use available stability estimates



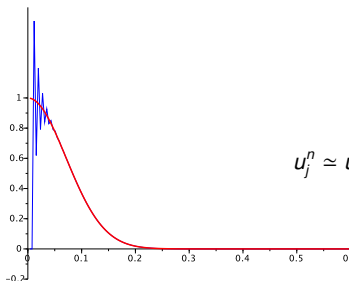
$$u_j^n \simeq u_{j,n}^{app} = u^{int}(x_j, t^n) + u^{bl}\left(\frac{x_j}{\Delta x}, t^n\right)$$

$u^{int}(x, t)$: smooth physical part of the solution

$u^{bl}(j, t)$: sawtoothed high-frequency pattern localized close to the boundary

Multiscale approximation of the discrete solution

- expected accuracy partially destroyed: non-existence of discrete traces
↔ defective consistency at the boundary
- + convergence still available far from the boundary
↔ stability in some sense
- ? improve the consistency analysis and use available stability estimates



$$u_j^n \simeq u_{j,n}^{\text{app}} = \textcolor{red}{u}^{\text{int}}(x_j, t^n) + \textcolor{blue}{u}^{\text{bl}}\left(\frac{x_j}{\Delta x}, t^n\right)$$

$\textcolor{red}{u}^{\text{int}}(x, t)$: smooth physical part of the solution

$\textcolor{blue}{u}^{\text{bl}}(j, t)$: sawtoothed high-frequency pattern localized close to the boundary

Relation with steady state solutions

Example of the Lax-Wendroff scheme

$$u_j^{n+1} = (Qu_n)_j := u_j^n + \underbrace{\frac{\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}_{=:(\nabla_{\text{dis}} u^n)_j \text{ (discrete space-gradient term)}}$$

Steady states

$$u = Qu \Leftrightarrow u \in \ker \nabla_{\text{dis}} \Leftrightarrow \forall j, u_j = \alpha 1^j + \underbrace{\beta \left(\frac{\lambda - 1}{\lambda + 1} \right)^j}_{\in \ell^2(\mathbb{N}) \text{ for } \lambda \in (0,1]}$$

Fact

Under the CFL condition $\lambda \in (0, 1]$, the LW scheme with homogeneous Dirichlet b.c. $u_0 = 0$ has the following bounded steady states ($\alpha \in \mathbb{C}$):

$$u_j = \alpha \left(1 - \left(\frac{\lambda - 1}{\lambda + 1} \right)^j \right), \quad j \geq 0.$$

→ This construction generalizes for many Cauchy stable schemes (discussed later) and is related to the normal modes $u_j^n = z^n k^j$ for $z = 1$.

First experiments, problematics and strategies

Stability theory for the initial boundary value problem

- Stability theory for the continuous hyperbolic IBVP

- GKS theory for discrete schemes

- Stability results for Dirichlet b.c

Dirichlet boundary layer and semigroup estimate for discrete IBVP

High order outflowing extrapolation

Stability theory for the continuous IBVP [BENZONI-GAVAGE & SERRE]

Linear hyperbolic Initial Boundary Value Problem

$$\partial_t u + A \partial_x u = F(x, t) \quad (x, t) \in \mathbb{R}_+^{(d)} \times \mathbb{R}_+, \quad u(x, t) \in \mathbb{R}^N$$

$$B u(0, t) = g(t) \quad t \in \mathbb{R}_+$$

$$u(x, 0) = f(x) \quad x \in \mathbb{R}_+^{(d)}$$

IVP (no boundary) stability $\Leftrightarrow$ **Hyperbolicity** $\sup_{\xi} \|\exp(i\xi A)\| \leq C$

BVP (no initial data) (strong) stability $\Leftrightarrow$ **Uniform Kreiss-Lopatinskii Condition**

- for $d = 1$: $\mathbb{R}^N = \text{Ker } B \oplus E_+(A)$, meaning the b.c. $Bu_+ = g - Bu_-$ is invertible,
- for $d > 1$: frequency-parameterized determinant bounded away from 0 (uniform resolvent inequality).

Definition (Strong stability for the BVP)

$$(f \equiv 0) \quad \exists C > 0, \forall \gamma > 0 \quad \gamma \|e^{-\gamma t} u\|^2 + \|e^{-\gamma t} u|_{x=0}\|^2 \leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} F\|^2 + \|e^{-\gamma t} g\|^2 \right)$$

Usefull tools:

- Laplace-in-time transform / normal mode analysis
- + Fourier-in-space transform for $d \geq 2$ space-dimension : $x \in \mathbb{R}_+^{(d)} := \mathbb{R}_+ \times \mathbb{R}^{d-1}$
- Use of weighted $L_t^{2,\gamma} L_x^2$ norms ($\gamma > 0$) adapted to Fourier-Laplace-Parseval identities

Stability theory for the continuous IBVP [BENZONI-GAVAGE & SERRE]

Linear hyperbolic Initial Boundary Value Problem

$$\partial_t u + A \partial_x u = F(x, t) \quad (x, t) \in \mathbb{R}_+^{(d)} \times \mathbb{R}_+, \quad u(x, t) \in \mathbb{R}^N$$

$$B u(0, t) = g(t) \quad t \in \mathbb{R}_+$$

$$u(x, 0) = f(x) \quad x \in \mathbb{R}_+^{(d)}$$

IVP (no boundary) stability $\Leftrightarrow$ **Hyperbolicity** $\sup_{\xi} \|\exp(i\xi A)\| \leq C$

BVP (no initial data) (strong) stability $\Leftrightarrow$ **Uniform Kreiss-Lopatinskii Condition**

- for $d = 1$: $\mathbb{R}^N = \text{Ker } B \oplus E_+(A)$, meaning the b.c. $Bu_+ = g - Bu_-$ is invertible,
- for $d > 1$: frequency-parameterized determinant bounded away from 0 (uniform resolvent inequality).

Definition (Strong stability for the BVP)

$$(f \equiv 0) \quad \exists C > 0, \forall \gamma > 0 \quad \gamma \|e^{-\gamma t} u\|^2 + \|e^{-\gamma t} u|_{x=0}\|^2 \leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} F\|^2 + \|e^{-\gamma t} g\|^2 \right)$$

Usefull tools:

- Laplace-in-time transform / normal mode analysis
- + Fourier-in-space transform for $d \geq 2$ space-dimension : $x \in \mathbb{R}_+^{(d)} := \mathbb{R}_+ \times \mathbb{R}^{d-1}$
- Use of weighted $L_t^{2,\gamma} L_x^2$ norms ($\gamma > 0$) adapted to Fourier-Laplace-Parseval identities

Stability theory for the continuous IBVP [BENZONI-GAVAGE & SERRE]

Strong stability implies semigroup stability (multidimensional case) :
(includes **nonzero initial data** f)

- FRIEDRICHS '67 for symmetric systems with dissipative b.c
- RAUCH '72 for strictly hyperbolic systems
- AUDIARD '11 for systems with constant multiplicities
- METIVIER' 14 for a more generalized class (block structure)

Tools: dissipative symmetrizers and energy multipliers, mimicing the classical energy method for symmetric systems with strictly dissipative b.c.

Theorem (Full stability estimate for the IBVP)

$$\begin{aligned} e^{-2\gamma T} \|u(\cdot, T)\|_{L^2(\mathbb{R}_+)}^2 + \gamma \|e^{-\gamma t} u\|_{L^2(\mathbb{R}_+ \times [0, T])}^2 + \|e^{-\gamma t} u|_{x=0}\|_{L^2([0, T])}^2 \\ \leq C \left(\|f\|_{L^2(\mathbb{R}_+)}^2 + \frac{1}{\gamma} \|e^{-\gamma t} F\|_{L^2(\mathbb{R}_+ \times [0, T])}^2 + \|e^{-\gamma t} g\|_{L^2([0, T])}^2 \right) \end{aligned}$$

In particular, for $g \equiv 0$ and $F \equiv 0$ one has $\|u(\cdot, T)\|_{L^2(\mathbb{R}_+)} \leq C(T) \|f\|_{L^2(\mathbb{R}_+)}.$

Stability theory for the discrete IBVP [GUSTAFSSON, KREISS, OLIGER]

Consider a numerical solution (u_j^n) , obtained from data F , g and f , formally say by

$$\begin{cases} u_j^{n+1} = (Qu^n)_j + \Delta t F_j^n & j \in \mathcal{D} \text{ (set of interior indices)} \\ (Bu^n)_j = g_j^n & j \in \partial \text{ (set of boundary indices)} \\ u_j^0 = f_j & j \in \mathcal{D} \end{cases}$$

Norm notations

$$\begin{aligned} \|u\|_{\mathcal{D}}^2 &= \sum_{j \in \mathcal{D}} \Delta x |u_j|^2, & \|u\|_{\partial}^2 &= \sum_{j \in \partial} |u_j|^2 \\ \|u\|_{\mathcal{D}, \gamma}^2 &= \sum_{n \geq 0} \Delta t e^{-2\gamma n \Delta t} \|u^n\|_{\mathcal{D}}^2, & \|u\|_{\partial, \gamma}^2 &= \sum_{n \geq 0} \Delta t e^{-2\gamma n \Delta t} \|u^n\|_{\partial}^2 \end{aligned}$$

Definition (Strong stability $\ell_t^{2, \gamma} \ell_x^2$, known as GKS stability^a)

$$(f \equiv 0) \quad \frac{\gamma}{1 + \gamma \Delta t} \|u\|_{\mathcal{D}, \gamma}^2 + \|u\|_{\partial, \gamma}^2 \leq C \left(\frac{1 + \gamma \Delta t}{\gamma} \|F\|_{\mathcal{D}, \gamma}^2 + \|g\|_{\partial, \gamma}^2 \right)$$

^a(GUSTAFSSON, KREISS & SUNDSTRÖM '72)

Stability theory for the discrete IBVP [GUSTAFSSON, KREISS, OLIGER]

Consider a numerical solution (u_j^n) , obtained from data F , g and f , formally say by

$$\begin{cases} u_j^{n+1} = (Qu^n)_j + \Delta t F_j^n & j \in \mathcal{D} \text{ (set of interior indices)} \\ (Bu^n)_j = g_j^n & j \in \partial \text{ (set of boundary indices)} \\ u_j^0 = f_j & j \in \mathcal{D} \end{cases}$$

Definition (Strong stability $\ell_t^{2,\gamma} \ell_x^2$, known as GKS stability^a)

$$(f \equiv 0) \quad \frac{\gamma}{1 + \gamma \Delta t} \|u\|_{\mathcal{D},\gamma}^2 + \|u\|_{\partial,\gamma}^2 \leq C \left(\frac{1 + \gamma \Delta t}{\gamma} \|F\|_{\mathcal{D},\gamma}^2 + \|g\|_{\partial,\gamma}^2 \right)$$

^a(GUSTAFSSON, KREISS & SUNDSTRÖM '72)

Strong stability is equivalent to an algebraic condition (UKLC)

Uniform (determinant) version of the [Godunov-Ryabenkii condition](#)

Preclude unstable modes $u_j^n = z^n \phi_j$ with $\phi \in \ell^2(\mathcal{D})$ and $|z| > 1$ ($F \equiv 0$ and $g \equiv 0$)

Instability of Difference Models for Hyperbolic Initial Boundary Value Problems

LLOYD N. TREFETHEN
Courant Institute

Growth rates in discrete IBVP

Table I
Summary of Unstable Growth Rates for Various Classes of Difference Models

	$\exists$ rightg st-st soln	$\exists$ rightg st-st soln	$\exists$ str rightg st-st soln	$\exists$ str rightg st-st soln with inf. r.c.	$\exists$ str rightg st-st soln with $ z > 1$ (Godunov-Ryabenkii)
growth wrt initial data	1	1 Thms 1a, 1b	$\sqrt{n}$ Thms 2a, 2b	n Thms 3a, 3b	const^n
growth wrt bndry data	1	$\sqrt{n}$ Thm 4b	n Thms 5a, 5b	n Thms 6a, 6b	const^n

Normal mode

$$u_j^n = z^n \kappa^j, j \geq 0.$$

Algebraic relation through the discrete dispersion relation

$$P(z, \kappa) = 0$$

to test against the boundary condition.

←→
GKS-stable GKS-unstable

Table III
Classification of Solutions (3.20) to Q into Leftgoing and Rightgoing Components

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$ z > 1$ $ \kappa > 1$	$ z = 1$ $ \kappa = 1$ $C < 0$	$ z = 1$ $ \kappa > 1$	$ z = 1$ $ \kappa = 1$ $C = 0$ $\delta = \nu_\ell$ $= \nu_r + 1$	$ z = 1$ $ \kappa = 1$ $C = 0$ $\delta \leq$ $\min\{\nu_\ell, \nu_r\}$	$ z = 1$ $ \kappa = 1$ $C = 0$ $\delta = \nu_r$ $= \nu_\ell + 1$	$ z = 1$ $ \kappa < 1$	$ z = 1$ $ \kappa = 1$ $C > 0$	$ z > 1$ $ \kappa < 1$
strictly leftgoing			strictly rightgoing					
leftgoing					rightgoing			

Another possible behavior: incoming/reflected wave packets

- ★ The leap frog scheme ($N = 300$)

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0, \quad 1 \leq j \leq N-1, \quad u_0^n = u_N^n = 0.$$

Bounded (ℓ^2 -stable) oscillating pattern:

$$u_j^n = z^n \kappa^j \text{ with } z = \kappa = \pm 1.$$

$$u_j^n \simeq 1^{j+n} u_{(1,1)}(x_j + t^n) + (-1)^{j+n} u_{(-1,-1)}(x_j - t^n).$$

Another possible behavior: incoming/reflected wave packets

- ★ The leap frog scheme ($N = 300$)

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0, \quad 1 \leq j \leq N-1, \quad u_0^n = u_N^n = 0.$$

Bounded (ℓ^2 -stable) oscillating
pattern:

$$u_j^n = z^n \kappa^j \text{ with } z = \kappa = \pm 1.$$

$$u_j^n \simeq 1^{j+n} u_{(1,1)}(x_j + t^n) + (-1)^{j+n} u_{(-1,-1)}(x_j - t^n).$$

Discrete semigroup stability results

An important result : GOLDBERG & TADMOR '81.

In the scalar case, considering the discrete **Dirichlet boundary condition**:
the discrete Cauchy stability implies the GKS stability.

Cauchy stability + GKS stability $\Rightarrow$ semigroup stability ($f \neq 0$)

- WU '95. Scalar equations or one-dimensional systems, one-step difference schemes.

Tool: by a superposition argument, design auxiliary strictly dissipative boundary conditions, and use the Goldberg-Tadmor result to connect with Dirichlet boundary condition.

- COULOMBEL & GLORIA '11. Extension to systems with several space dimensions and variable coefficients, one-step difference schemes.

Tool: energy method and another auxiliary dissipative boundary conditions, without using the GKS stability result.

- COULOMBEL '15. Multistep multidimensional systems. Assuming the simplicity of roots in the von Neumann Cauchy stability

Tool: Leray-Gårding multipliers, auxiliary strictly dissipative boundary condition.

- B. & COULOMBEL '17. Multistep onedimensional scheme. Assuming the absence of incoming or glancing discrete wave packets

Tool: Constructive discrete boundary layer expansion and Goldberg-Tadmor lemma.

Discrete semigroup stability results

An important result : GOLDBERG & TADMOR '81.

In the scalar case, considering the discrete **Dirichlet boundary condition**:
the discrete Cauchy stability implies the GKS stability.

Cauchy stability + GKS stability $\Rightarrow$ semigroup stability ($f \neq 0$)

- WU '95. Scalar equations or one-dimensional systems, one-step difference schemes.

Tool: by a superposition argument, design auxiliary strictly dissipative boundary conditions, and use the Goldberg-Tadmor result to connect with Dirichlet boundary condition.

- COULOMBEL & GLORIA '11. Extension to systems with several space dimensions and variable coefficients, one-step difference schemes.

Tool: energy method and another auxiliary dissipative boundary conditions, without using the GKS stability result.

- COULOMBEL '15. Multistep multidimensional systems. Assuming the simplicity of roots in the von Neumann Cauchy stability

Tool: Leray-Gårding multipliers, auxiliary strictly dissipative boundary condition.

- B. & COULOMBEL '17. Multistep onedimensional scheme. Assuming the absence of incoming or glancing discrete wave packets

Tool: Constructive discrete boundary layer expansion and Goldberg-Tadmor lemma.

Discrete semigroup stability results

An important result : GOLDBERG & TADMOR '81.

In the scalar case, considering the discrete **Dirichlet boundary condition**:
the discrete Cauchy stability implies the GKS stability.

Cauchy stability + GKS stability $\Rightarrow$ semigroup stability ($f \neq 0$)

- WU '95. Scalar equations or one-dimensional systems, **one-step difference schemes**.

Tool: by a superposition argument, design auxiliary strictly dissipative boundary conditions, and use the Goldberg-Tadmor result to connect with Dirichlet boundary condition.

- COULOMBEL & GLORIA '11. Extension to systems with several space dimensions and variable coefficients, **one-step difference schemes**.

Tool: energy method and another auxiliary dissipative boundary conditions, without using the GKS stability result.

- COULOMBEL '15. Multistep multidimensional systems. **Assuming the simplicity of roots in the von Neumann Cauchy stability**

Tool: Leray-Gårding multipliers, auxiliary strictly dissipative boundary condition.

- B. & COULOMBEL '17. Multistep onedimensional scheme. **Assuming the absence of incoming or glancing discrete wave packets**

Tool: Constructive discrete boundary layer expansion and Goldberg-Tadmor lemma.

First experiments, problematics and strategies

Stability theory for the initial boundary value problem

Dirichlet boundary layer and semigroup estimate for discrete IBVP

- Multistep MOL schemes

- Solving the boundary layer

- Numerical experiments

- Semigroup estimate

High order outflowing extrapolation

Multistep Method Of Lines schemes

Scalar one-dimensional transport equation ($a < 0$ or $a > 0$)

$$\begin{aligned}\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} &= 0, & (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(x, 0) &= f(x), & x \in \mathbb{R}^+ \\ u_+(0, t) &= 0, & t \in \mathbb{R}^+\end{aligned}$$

Explicit multistep "Method Of Lines" finite difference schemes

$u_j^n \approx u(j\Delta x, n\Delta t)$, CFL parameter $\lambda = \Delta t / \Delta x$

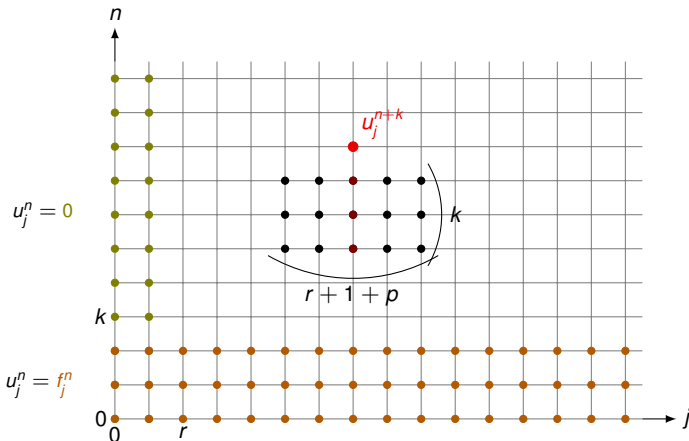
$$\boxed{\sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0} \quad r \leq j \quad 0 \leq n$$

$$u_j^n = f_j^n := f(x_j - at^n) \quad 0 \leq j \quad 0 \leq n \leq k-1$$

$$u_j^n = 0 \quad 0 \leq j \leq r-1 \quad k \leq n$$

Numerical stencil and notations

$$\sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0, \quad r \leq j, \quad 0 \leq n$$



Strategy : consistency analysis with improved description

- Find an expansion $u_{j,n}^{\text{app}} = u_j^n + e_{j,n}$ such that
 - $u_{j,n}^{\text{app}}$ is a **sufficiently accurate** description of u_j^n using the boundary layer,
 - so that $e_{j,n}$, the residual error terms, solves the discrete IBVP with **zero initial data** and **small boundary terms** and **small source terms**.

$$\sum_{\sigma=0}^k \alpha_{\sigma} e_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} e_{j+\ell,n+\sigma} = \Delta t \varepsilon_{j,n+k}$$

$$e_{j,n} = \eta_{j,n}, \quad (0 \leq j \leq r-1)$$

$$e_{j,0} = \cdots = e_{j,k-1} = 0, \quad (j \geq 0)$$

$$\text{where we set } \begin{cases} \varepsilon_{j,n+k} := \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{app}} \right) \\ \eta_{j,n} := u_{j,n}^{\text{app}} \end{cases}$$

- Goldberg-Tadmor lemma applied to $e_{j,n}$ gives GKS strong estimate
- Get error estimates for $\varepsilon_{j,n+k}$ and $\eta_{j,n} \Rightarrow$ semigroup estimate for $e_{j,n}$
- Get semigroup estimate on $u_{j,n}^{\text{app}}$ from the expression (1.) of

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

in terms of the initial data f

Strategy : consistency analysis with improved description

- Find an expansion $u_{j,n}^{\text{app}} = u_j^n + e_{j,n}$ such that
 - $u_{j,n}^{\text{app}}$ is a **sufficiently accurate** description of u_j^n using the boundary layer,
 - so that $e_{j,n}$, the residual error terms, solves the discrete IBVP with **zero initial data** and **small boundary terms** and **small source terms**.

$$\sum_{\sigma=0}^k \alpha_{\sigma} e_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} e_{j+\ell,n+\sigma} = \Delta t \varepsilon_{j,n+k}$$

$$e_{j,n} = \eta_{j,n}, \quad (0 \leq j \leq r-1)$$

$$e_{j,0} = \cdots = e_{j,k-1} = 0, \quad (j \geq 0)$$

- Goldberg-Tadmor lemma** applied to $e_{j,n}$ gives GKS strong estimate
- Get **error estimates** for $\varepsilon_{j,n+k}$ and $\eta_{j,n} \Rightarrow$ semigroup estimate for $e_{j,n}$

- Get semigroup estimate on $u_{j,n}^{\text{app}}$ from the expression (1.) of

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

in terms of the initial data f

- Results 3. and 4. $\Rightarrow$ semigroup estimate for u_j^n

Strategy : consistency analysis with improved description

- Find an expansion $u_{j,n}^{\text{app}} = u_j^n + e_{j,n}$ such that
 - $u_{j,n}^{\text{app}}$ is a **sufficiently accurate** description of u_j^n using the boundary layer,
 - so that $e_{j,n}$, the residual error terms, solves the discrete IBVP with **zero initial data** and **small boundary terms** and **small source terms**.

$$\sum_{\sigma=0}^k \alpha_{\sigma} e_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} e_{j+\ell,n+\sigma} = \Delta t \varepsilon_{j,n+k}$$

$$e_{j,n} = \eta_{j,n}, \quad (0 \leq j \leq r-1)$$

$$e_{j,0} = \cdots = e_{j,k-1} = 0, \quad (j \geq 0)$$

- Goldberg-Tadmor lemma** applied to $e_{j,n}$ gives GKS strong estimate
- Get **error estimates** for $\varepsilon_{j,n+k}$ and $\eta_{j,n} \Rightarrow$ semigroup estimate for $e_{j,n}$
- Get semigroup estimate on $u_{j,n}^{\text{app}}$ from the expression (1.) of

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

in terms of the initial data f

- Results 3. and 4. $\Rightarrow$ semigroup estimate for u_j^n

Reminder of the pure discrete Cauchy problem

$$\sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0 \quad \text{set over } j \in \mathbb{Z}$$

Fourier multiplier of the space discretization:

$$\mathcal{A}(\kappa) = \sum_{\ell=-r}^p a_{\ell} \kappa^{\ell}, \quad \kappa \neq 0$$

Linear recurrence relation of the multistep time discretization:

Characteristic polynomial: $P_{\mu}(X) = \rho(X) - \mu\sigma(X), \quad \mu \in \mathbb{C} (\leftarrow -\lambda \mathcal{A}(\kappa))$

with the two Dahlquist's generating polynomials:

$$\rho(X) = \sum_{\sigma=0}^k \alpha_{\sigma} X^{\sigma}, \quad \sigma(X) = \sum_{\sigma=0}^{k-1} \beta_{\sigma} X^{\sigma}$$

Consistency of the numerical scheme

Consistency at order (1,1)

$$\mathcal{A}(1) = 0, \quad \mathcal{A}'(1) = a,$$

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \quad (= 1 \text{ as normalization}).$$

Consistency at order (p,q)

$$\mathcal{A}(e^{i\xi}) - ia\xi = O(\xi^{q+1}),$$

$$\rho(e^h) - h\sigma(e^h) = O(h^{p+1}).$$

Reminder of the pure discrete Cauchy problem

$$\sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0 \quad \text{set over } j \in \mathbb{Z}$$

Fourier multiplier of the space discretization:

$$\mathcal{A}(\kappa) = \sum_{\ell=-r}^p a_{\ell} \kappa^{\ell}, \quad \kappa \neq 0$$

Linear recurrence relation of the multistep time discretization:

Characteristic polynomial: $P_{\mu}(X) = \rho(X) - \mu\sigma(X), \quad \mu \in \mathbb{C}(\leftarrow -\lambda \mathcal{A}(\kappa))$

with the two Dahlquist's generating polynomials:

$$\rho(X) = \sum_{\sigma=0}^k \alpha_{\sigma} X^{\sigma}, \quad \sigma(X) = \sum_{\sigma=0}^{k-1} \beta_{\sigma} X^{\sigma}$$

Consistency of the numerical scheme

Consistency at order (1,1)

$$\mathcal{A}(1) = 0, \quad \mathcal{A}'(1) = a,$$

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \quad (= 1 \text{ as normalization}).$$

Consistency at order (p,q)

$$\mathcal{A}(e^{i\xi}) - ia\xi = O(\xi^{q+1}),$$

$$\rho(e^h) - h\sigma(e^h) = O(h^{p+1}).$$

Reminder of the pure discrete Cauchy problem

$$\sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0 \quad \text{set over } j \in \mathbb{Z}$$

Fourier multiplier of the space discretization:

$$\mathcal{A}(\kappa) = \sum_{\ell=-r}^p a_{\ell} \kappa^{\ell}, \quad \kappa \neq 0$$

Linear recurrence relation of the multistep time discretization:

Characteristic polynomial: $P_{\mu}(X) = \rho(X) - \mu\sigma(X), \quad \mu \in \mathbb{C} (\leftarrow -\lambda \mathcal{A}(\kappa))$

with the two Dahlquist's generating polynomials:

$$\rho(X) = \sum_{\sigma=0}^k \alpha_{\sigma} X^{\sigma}, \quad \sigma(X) = \sum_{\sigma=0}^{k-1} \beta_{\sigma} X^{\sigma}$$

Consistency of the numerical scheme

Consistency at order (1,1)

$$\mathcal{A}(1) = 0, \quad \mathcal{A}'(1) = a,$$

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \quad (= 1 \text{ as normalization}).$$

Consistency at order (p,q)

$$\mathcal{A}(e^{i\xi}) - ia\xi = O(\xi^{q+1}),$$

$$\rho(e^h) - h\sigma(e^h) = O(h^{p+1}).$$

Stability theory for the discrete Cauchy problem

The power boundedness of the companion matrices in the time recurrence relation define the **Stability region** of the multistep method (Ref. Hairer-Wanner.)

$$\mathcal{S} = \left\{ \mu \in \mathbb{C}, P_\mu(z) = 0 \Rightarrow \begin{cases} |z| < 1, \text{ or} \\ |z| = 1 \text{ and } z \text{ is simple} \end{cases} \right\}.$$

Common theorem

The numerical scheme is (Cauchy-) ℓ^2 stable iff $|k| = 1 \Rightarrow -\mathcal{A}(k) \in \mathcal{S}$.

From now on, the considered scheme is assumed to satisfy these assumptions and so

$$\exists C > 0, \forall \Delta t \in (0, 1), \forall (f^\sigma)_{0 \leq \sigma \leq k-1} \in (\ell_x^2(\mathbb{Z}))^k$$

$$\sup_{n \geq 0} \|u^n\|_{\ell^2(\mathbb{Z})} \leq C \sum_{\sigma=0}^{k-1} \|f_j^\sigma\|_{\ell^2(\mathbb{Z})}$$

Stability theory for the discrete Cauchy problem

The power boundedness of the companion matrices in the time recurrence relation define the **Stability region** of the multistep method (Ref. Hairer-Wanner.)

$$\mathcal{S} = \left\{ \mu \in \mathbb{C}, P_{\mu}(z) = 0 \Rightarrow \begin{cases} |z| < 1, \text{ or} \\ |z| = 1 \text{ and } z \text{ is simple} \end{cases} \right\}.$$

Common theorem

The numerical scheme is (Cauchy-) ℓ^2 stable iff $|k| = 1 \Rightarrow -\lambda \mathcal{A}(k) \in \mathcal{S}$.

From now on, the considered scheme is assumed to satisfy these assumptions and so

$$\exists C > 0, \forall \Delta t \in (0, 1), \forall (f^{\sigma})_{0 \leq \sigma \leq k-1} \in \left(\ell_x^2(\mathbb{Z}) \right)^k$$

$$\sup_{n \geq 0} \|u^n\|_{\ell^2(\mathbb{Z})} \leq C \sum_{\sigma=0}^{k-1} \|f_{\sigma}^{\sigma}\|_{\ell^2(\mathbb{Z})}$$

Stability theory for the discrete Cauchy problem

The power boundedness of the companion matrices in the time recurrence relation define the **Stability region** of the multistep method (Ref. Hairer-Wanner.)

$$\mathcal{S} = \left\{ \mu \in \mathbb{C}, P_\mu(z) = 0 \Rightarrow \begin{cases} |z| < 1, \text{ or} \\ |z| = 1 \text{ and } z \text{ is simple} \end{cases} \right\}.$$

Common theorem

The numerical scheme is (Cauchy-) ℓ^2 stable iff $|k| = 1 \Rightarrow -\lambda \mathcal{A}(k) \in \mathcal{S}$.

From now on, the considered scheme is assumed to satisfy these assumptions and so

$$\exists C > 0, \forall \Delta t \in (0, 1), \forall (f^\sigma)_{0 \leq \sigma \leq k-1} \in (\ell_x^2(\mathbb{Z}))^k$$

$$\sup_{n \geq 0} \|u^n\|_{\ell^2(\mathbb{Z})} \leq C \sum_{\sigma=0}^{k-1} \|f_j^\sigma\|_{\ell^2(\mathbb{Z})}$$

Example of the AB3-5pts scheme

$$u_j^{n+1} = u_j^n - \lambda \left(\frac{23}{12} v_j^n - \frac{16}{12} v_j^{n-1} + \frac{5}{12} v_j^{n-2} \right)$$

$$v_j^n := a \frac{-u_{j+2}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-2}^n}{12} - \frac{-u_{j+2}^n + 4u_{j+1}^n - 6u_j^n + 4u_{j-1}^n - u_{j-2}^n}{24}$$

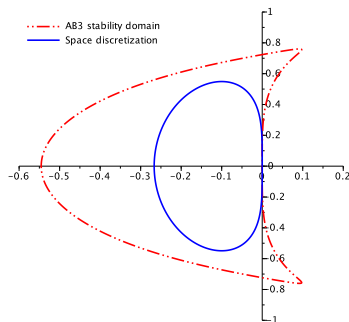
$$P_\mu(X) = X^3 - X^2 - \mu \left(\frac{23}{12} X^2 - \frac{16}{12} X + \frac{5}{12} \right)$$

$$\mathcal{A}(\kappa) = \frac{a}{12} (-\kappa^2 + 8\kappa - 8\kappa^{-1} + \kappa^{-2}) - \frac{1}{24} (-\kappa^2 + 4\kappa - 6 + 4\kappa^{-1} - \kappa^{-2})$$

Stability assumption:

(CFL parameter $\lambda = 0.4$)

Consistent at order (3,3).



① Far from the boundary: u^{int}

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{app}} \right)$$

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

Scale x

Fix $x \in \mathbb{R}_+^*$, let $\Delta t, \Delta x \rightarrow 0$.

For $j \rightarrow \infty$ but $x_j \simeq x$: $u^{\text{bl}}(j, t^n) \simeq 0$ and $u_{j,n}^{\text{app}} \simeq u_{j,n}^{\text{int}}$.

Therefore

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{int}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{int}} \right).$$

We introduce $u^{\text{int}}(x, t)$ as the solution of the unbounded domain problem:

$$\partial_t u + a \partial_x u = 0, \quad x \in \mathbb{R}, \quad t \geq 0$$

$$u(x, 0) = f(x) \mathbb{1}_{\mathbb{R}_+} + 0 \times \mathbb{1}_{\mathbb{R}_-}$$

① Far from the boundary: u^{int}

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{app}} \right)$$

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

Scale x

Fix $x \in \mathbb{R}_+^*$, let $\Delta t, \Delta x \rightarrow 0$.

For $j \rightarrow \infty$ but $x_j \simeq x$: $u^{\text{bl}}(j, t^n) \simeq 0$ and $u_{j,n}^{\text{app}} \simeq u_{j,n}^{\text{int}}$.

Therefore

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{int}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{int}} \right).$$

We introduce $u^{\text{int}}(x, t)$ as the solution of the unbounded domain problem:

$$\partial_t u + a \partial_x u = 0, \quad x \in \mathbb{R}, \quad t \geq 0$$

$$u(x, 0) = f(x) \mathbf{1}_{\mathbb{R}_+} + 0 \times \mathbf{1}_{\mathbb{R}_-}$$

② Leading boundary layer profile: $u^{\text{bl},0}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{app}} \right)$$

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

Scale Δx

Fix $j \in \mathbb{Z}$, let $\Delta t, \Delta x \rightarrow 0$.

For u^{int} smooth: $u^{\text{int}}(x_{j+\ell}, t^{n+\sigma}) \simeq u^{\text{int}}(0, t^n)$

Expected time-regularity of the boundary layer: $u^{\text{bl}}(j, t^{n+\sigma}) \simeq u^{\text{bl}}(j, t^n)$.

Therefore

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta x} \left(\sum_{\sigma=0}^{k-1} \beta_{\sigma} \right) \sum_{\ell=-r}^p a_{\ell} u^{\text{bl}}(j + \ell, t^{n+k}).$$

We require $(u^{\text{bl}}(j, t))_j$ to satisfy:

red term=0

b.c. $u^{\text{bl}}(j, t) + u^{\text{int}}(0, t) = 0, \quad 0 \leq j \leq r-1,$

localization property $\lim_{j \rightarrow \infty} u^{\text{bl}}(j, t) = 0$.

② Leading boundary layer profile: $u^{\text{bl},0}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t} \left(\sum_{\sigma=0}^k \alpha_{\sigma} u_{j,n+\sigma}^{\text{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell,n+\sigma}^{\text{app}} \right)$$

$$u_{j,n}^{\text{app}} = u^{\text{int}}(x_j, t^n) + u^{\text{bl}}(j, t^n)$$

Scale Δx

Fix $j \in \mathbb{Z}$, let $\Delta t, \Delta x \rightarrow 0$.

For u^{int} smooth: $u^{\text{int}}(x_{j+\ell}, t^{n+\sigma}) \simeq u^{\text{int}}(0, t^n)$

Expected time-regularity of the boundary layer: $u^{\text{bl}}(j, t^{n+\sigma}) \simeq u^{\text{bl}}(j, t^n)$.

Therefore

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta x} \left(\sum_{\sigma=0}^{k-1} \beta_{\sigma} \right) \sum_{\ell=-r}^p a_{\ell} u^{\text{bl}}(j + \ell, t^{n+k}).$$

We require $(u^{\text{bl}}(j, t))_j$ to satisfy:

red term=0

b.c. $u^{\text{bl}}(j, t) + u^{\text{int}}(0, t) = 0, \quad 0 \leq j \leq r-1,$

localization property $\lim_{j \rightarrow \infty} u^{\text{bl}}(j, t) = 0$.

Boundary layer definition

Definition

Being given $u \in \mathbb{R}$, a sequence $(v_j)_{j \in \mathbb{N}}$ is said to be a **boundary layer profile** associated with u if:

1. $v_0 = \dots = v_{r-1} = -u$,
2. $\sum_{\ell=-r}^p a_\ell v_{j+\ell+r} = 0$ for all $j \geq 0$,
3. $\lim_{j \rightarrow \infty} v_j = 0$.

Denote C_{num} the set of all u such that a boundary layer exists.

- ? Identify the set C_{num} ?
- ? Being given $u \in C_{\text{num}}$, is there a unique associated **boundary layer profile** ?

Related litterature for the nonlinear case

DUBOIS & LEFLOCH '88 - admissible entropy boundary data (PDE)

GISCLON & SERRE '97 - residual boundary conditions for the Godunov scheme

CHAINAIS-HILLAIRET & GRENIER '01 - conservative schemes

Solving the boundary layer

Roots of $\mathcal{A}(\kappa) = \sum_{\ell=-r}^p a_{\ell} \kappa^{\ell}$

- consistency root $\kappa = 1 \leftrightarrow$ physical solution travelling at velocity a
- roots κ with $|\kappa| < 1 \leftrightarrow$ boundary layer terms localized at $j = 0$
- ⚡ roots with $|\kappa| = 1 \leftrightarrow$ possible unphysical travelling/glancing wave packet profiles

Assumption (H)

$\kappa = 1$ is the unique root of $\mathcal{A}$ with $|\kappa| = 1$

Examples

- ★ Explicit Euler time discretization: $P_{\mu}(X) = X - 1 - \mu$

$$\mathcal{A}(e^{i\eta}) = 0 \Leftrightarrow 1 - \lambda \mathcal{A}(e^{i\eta}) = 1$$

- Any dissipative scheme satisfies (H):

$$\exists c > 0, \exists m \in \mathbb{N}^*, \forall |\eta| \leq \pi, |1 - \lambda \mathcal{A}(e^{i\eta})| \leq 1 - c\eta^{2m}.$$

- Some other common non-dissipative schemes also satisfy (H).

$$\text{The Lax-Friedrichs : } u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda a}{2}(u_{j+1}^n - u_{j-1}^n),$$

$$\text{with } 1 - \lambda \mathcal{A}(e^{i\eta}) = \cos \eta - i\lambda a \sin \eta.$$

- ★ The leap-frog scheme does not : $\mathcal{A}(-1) = 0$.

- ★ The proposed AB3 - 5pts scheme also satisfies (H) : $\Re \mathcal{A}(e^{i\eta}) = \frac{2}{3} \sin^4\left(\frac{\eta}{2}\right)$.

Roots: $\kappa_1 \approx -0.6595$, $\kappa_2 \approx 0.0809$ depend on CFL only

Solving the boundary layer

Roots of $\mathcal{A}(\kappa) = \sum_{\ell=-r}^p a_{\ell} \kappa^{\ell}$

- consistency root $\kappa = 1 \leftrightarrow$ physical solution travelling at velocity a
- roots κ with $|\kappa| < 1 \leftrightarrow$ boundary layer terms localized at $j = 0$
- ⚡ roots with $|\kappa| = 1 \leftrightarrow$ possible unphysical travelling/glancing wave packet profiles

Assumption (H)

$\kappa = 1$ is the unique root of $\mathcal{A}$ with $|\kappa| = 1$

Examples

- ★ Explicit Euler time discretization: $P_{\mu}(X) = X - 1 - \mu$

$$\mathcal{A}(e^{i\eta}) = 0 \Leftrightarrow 1 - \lambda \mathcal{A}(e^{i\eta}) = 1$$

- Any **dissipative** scheme satisfies (H):

$$\exists c > 0, \exists m \in \mathbb{N}^*, \forall |\eta| \leq \pi, |1 - \lambda \mathcal{A}(e^{i\eta})| \leq 1 - c\eta^{2m}.$$

- Some other common non-dissipative schemes also satisfy (H).

$$\text{The Lax-Friedrichs : } u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda a}{2}(u_{j+1}^n - u_{j-1}^n),$$

$$\text{with } 1 - \lambda \mathcal{A}(e^{i\eta}) = \cos \eta - i\lambda a \sin \eta.$$

- ★ The leap-frog scheme does not : $\mathcal{A}(-1) = 0$.

- ★ The proposed AB3 - 5pts scheme also satisfies (H) : $\Re \mathcal{A}(e^{i\eta}) = \frac{2}{3} \sin^4 \left(\frac{\eta}{2} \right)$.

Roots: $\kappa_1 \simeq -0.6595$, $\kappa_2 \simeq 0.0809$ depend on CFL only

Solving the boundary layer

Let $|z| > 1$. Cauchy stability: no roots $|\kappa| = 1$ of $\rho(z) + \lambda\sigma(z)\mathcal{A}(\kappa) = 0$
Continuity and separation of roots $\Rightarrow$ exactly r roots with $|\kappa| < 1$
and p roots with $|\kappa| > 1$ ("Hersh lemma")

Lemma (Counting roots for $z = 1$)

*Under the Cauchy stability assumption and the above assumption (H),
the function $\mathcal{A}$ admits exactly R roots (with multiplicity) in $\{\kappa \in \mathbb{C}, 0 < |\kappa| < 1\}$*

where $R = r$ (for $a < 0$) or $r - 1$ (for $a > 0$).

Solving the boundary layer

Let $|z| > 1$. Cauchy stability: no roots $|\kappa| = 1$ of $\rho(z) + \lambda\sigma(z)\mathcal{A}(\kappa) = 0$
 Continuity and separation of roots $\Rightarrow$ exactly r roots with $|\kappa| < 1$
 and p roots with $|\kappa| > 1$ ("Hersh lemma")

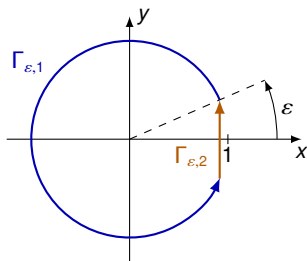
Lemma (Counting roots for $z = 1$)

Under the Cauchy stability assumption and the above assumption (H),
 the function $\mathcal{A}$ admits exactly R roots (with multiplicity) in $\{\kappa \in \mathbb{C}, 0 < |\kappa| < 1\}$

where $R = r$ (for $a < 0$) or $r - 1$ (for $a > 0$).

Proof.
$$\frac{1}{2i\pi} \int_{\Gamma} \frac{\mathcal{A}'(\kappa)}{\mathcal{A}(\kappa)} d\kappa = \#\{\text{zeros}\} - \#\{\text{poles}\},$$

- 0 is pole of order r
- 1 is zero of order 1 : $\mathcal{A}(1) = 0, \mathcal{A}'(1) = a \neq 0$
- does not vanish on $\Gamma_{\epsilon,1}$, therefore^a $\mathcal{A}(\kappa) \notin \mathbb{R}_-^*$ / use $\log_-$
- $a\mathcal{A}(\kappa) \notin \mathbb{R}_+$ for $\kappa \in \Gamma_{\epsilon,2}$ (ϵ being sufficiently small):
 - case $a < 0$: $\mathcal{A}(\kappa) \notin \mathbb{R}_-$ for $\kappa \in \Gamma_{\epsilon,2}$ / use $\log_-$: $R = r$
 - case $a > 0$: $\mathcal{A}(\kappa) \notin \mathbb{R}_+$ for $\kappa \in \Gamma_{\epsilon,2}$ / use $\log_+$: $R = r - 1$.



^aThe stability region $\mathcal{S}$ contains no positive real number

Solving the boundary layer

Let $|z| > 1$. Cauchy stability: no roots $|\kappa| = 1$ of $\rho(z) + \lambda\sigma(z)\mathcal{A}(\kappa) = 0$
 Continuity and separation of roots $\Rightarrow$ exactly r roots with $|\kappa| < 1$
 and p roots with $|\kappa| > 1$ ("Hersh lemma")

Lemma (Counting roots for $z = 1$)

Under the Cauchy stability assumption and the above assumption (H),
 the function $\mathcal{A}$ admits exactly R roots (with multiplicity) in $\{\kappa \in \mathbb{C}, 0 < |\kappa| < 1\}$

where $R = r$ (for $a < 0$) or $r - 1$ (for $a > 0$).

Canonical boundary layer profile adapted to the Dirichlet b.c for $a < 0$

$$\text{(assume the roots are simple)} \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ \kappa_1 & \kappa_2 & \dots & \kappa_r \\ \vdots & \vdots & & \vdots \\ \kappa_1^{r-1} & \kappa_2^{r-1} & \dots & \kappa_r^{r-1} \end{pmatrix} \begin{pmatrix} z_1(t) \\ \vdots \\ z_r(t) \end{pmatrix} + u(0, t) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

$$u^{\text{bl}}(j, t) = \sum_{\ell=1}^r z_{\ell}(t) \kappa_{\ell}^j = u(0, t) \sum_{\ell=1}^r \alpha_{\ell} \kappa_{\ell}^j$$

Summary of the boundary layer expansion

$$u_j^n \simeq u_{j,n}^{\text{app}} := u_{j,n}^{\text{int}} + u_{j,n}^{\text{bl},0} + \Delta x u_{j,n}^{\text{bl},1}$$

- ① Internal layer (expected exact solution)

$$u_{j,n}^{\text{int}} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} f(x - at^n) dx.$$

- ② Leading boundary layer

$$u_{j,n}^{\text{bl},0} = u_n^{\text{tr}} w_j, \quad j \geq 0, \quad n \geq k,$$

where the numerical trace at time t^n reads

$$u_n^{\text{tr}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(0 - at) dt.$$

- ③ First boundary layer corrector (for time variations of $u^{\text{bl},0}$)

$$u_{j,n}^{\text{bl},1} = (\Delta t \sum \beta_\sigma)^{-1} \left(\sum \alpha_\sigma u_{n+\sigma}^{\text{tr}} \right) \widetilde{w}_j, \quad j \geq 0, \quad n \geq k.$$

Experiments involving the boundary layer expansion

Comparisons

$$u_j^n \simeq u_{j,n}^{\text{app}} := u_{j,n}^{\text{int}} + u_{j,n}^{\text{bl},0} + \Delta x u_{j,n}^{\text{bl},1}$$

where (with some approximations for convenience in the presentation)

$$u_{j,n}^{\text{int}} = u(x_j, t^n) = f(x_j - at^n),$$

$$u_{j,n}^{\text{bl},0} = (\alpha_1 \kappa_1^j + \alpha_2 \kappa_2^j) u(0, t^n),$$

$$u_{j,n}^{\text{bl},1} = (P_1(j) \kappa_1^j + P_2(j) \kappa_2^j) \frac{\partial u}{\partial t}(0, t^n),$$

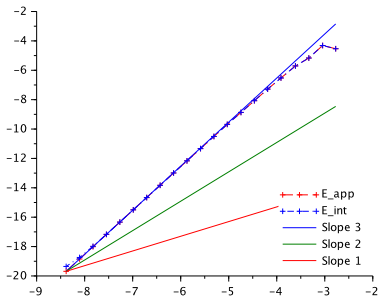
$$\begin{pmatrix} 1 & 1 \\ \kappa_1 & \kappa_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution computed at time $T = 0.4$ with different Δx

Experiments involving the boundary layer expansion

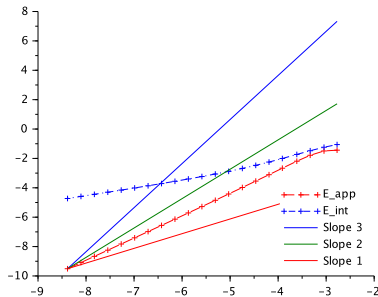
Rate of convergence ℓ^2

$$E_2^{\text{int}} := \left(\sum_{j=0}^J \Delta x \left| u_j^n - u^{\text{int}}(x_j, t^n) \right|^2 \right)^{1/2}, \quad E_2^{\text{app}} := \left(\sum_{j=0}^J \Delta x \left| u_j^n - u^{\text{app}}(x_j, t^n) \right|^2 \right)^{1/2}.$$



T=0.125

$$E_2^{\text{int}} = O(\Delta x^3), \quad E_2^{\text{app}} = O(\Delta x^3)$$



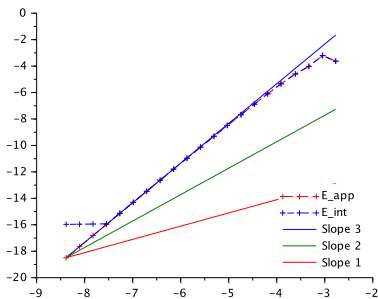
T=0.4

$$E_2^{\text{int}} = O(\Delta x^{1/2}), \quad E_2^{\text{app}} = O(\Delta x^{3/2})$$

Experiments involving the boundary layer expansion

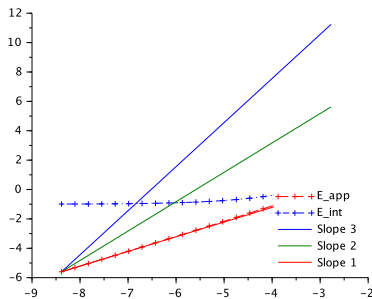
Rate of convergence ℓ^∞

$$E_\infty^{\text{int}} := \max_{0 \leq j \leq J} |u_j^n - u^{\text{int}}(x_j, t^n)|, \quad E_\infty^{\text{app}} := \max_{0 \leq j \leq J} |u_j^n - u^{\text{app}}(x_j, t^n)|.$$



$T=0.125$

$$E_\infty^{\text{int}} = O(\Delta x^3), \quad E_\infty^{\text{app}} = O(\Delta x^1)$$



$T=0.4$

$$E_\infty^{\text{int}} = O(\Delta x^0), \quad E_\infty^{\text{app}} = O(\Delta x^1)$$

Convergence analysis

Some ingredients:

- Usual consistency of the interior scheme
- Compatibility condition ($a < 0$) : homogeneous Dirichlet/initial data
- Exponential decrease in space of the boundary layer
- Consistency of the several approximations of the trace at the boundary

"GKS estimate" for the consistency error analysis

$$\forall \Delta t \in (0, 1], \forall \gamma > 0, \forall f \in H_0^2(\mathbb{R}^+) :$$

$$\|\epsilon\|_{\mathcal{D}, \gamma}^2 \lesssim \left(1 + \frac{1}{\gamma}\right) \Delta t^2 \|f\|_{H^2(\mathbb{R}^+)}^2, \quad \|\eta\|_{\partial, \gamma}^2 \lesssim \Delta t^2 \|f\|_{H^1(\mathbb{R}^+)}^2.$$

Strong stability result for Dirichlet (and zero initial data):

$$\frac{\gamma}{1 + \gamma \Delta t} \|e\|_{\mathcal{D}, \gamma}^2 + \|e\|_{\partial, \gamma}^2 \lesssim \frac{1 + \gamma \Delta t}{\gamma} \|\epsilon\|_{\mathcal{D}, \gamma}^2 + \|\eta\|_{\partial, \gamma}^2$$

Let $\gamma = \Delta t^\mu$ with $\mu > 0$

$$\sum_{n \geq 0} \Delta t e^{-2n\Delta t^{1+\mu}} \|e^n\|_{\mathcal{D}}^2 \lesssim \Delta t^{2-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2.$$

$$\sup_{n\Delta t \leq T} \|e^n\|_{\mathcal{D}}^2 \lesssim \sup_{n\Delta t \leq T} e^{2n\Delta t^{1+\mu}} \Delta t^{1-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2 \lesssim \Delta t T^3 \|f\|_{H^2(\mathbb{R}^+)}^2.$$

Convergence analysis

Some ingredients:

- Usual consistency of the interior scheme
- Compatibility condition ($a < 0$) : homogeneous Dirichlet/initial data
- Exponential decrease in space of the boundary layer
- Consistency of the several approximations of the trace at the boundary

"GKS estimate" for the consistency error analysis

$$\forall \Delta t \in (0, 1], \forall \gamma > 0, \forall f \in H_0^2(\mathbb{R}^+) :$$

$$\|e\|_{\mathcal{D},\gamma}^2 \lesssim \left(1 + \frac{1}{\gamma}\right) \Delta t^2 \|f\|_{H^2(\mathbb{R}^+)}^2, \quad \|\eta\|_{\partial,\gamma}^2 \lesssim \Delta t^2 \|f\|_{H^1(\mathbb{R}^+)}^2.$$

Strong stability result for Dirichlet (and zero initial data):

$$\frac{\gamma}{1 + \gamma \Delta t} \|e\|_{\mathcal{D},\gamma}^2 + \|e\|_{\partial,\gamma}^2 \lesssim \frac{1 + \gamma \Delta t}{\gamma} \|e\|_{\mathcal{D},\gamma}^2 + \|\eta\|_{\partial,\gamma}^2$$

Let $\gamma = \Delta t^\mu$ with $\mu > 0$

$$\sum_{n \geq 0} \Delta t e^{-2n\Delta t^{1+\mu}} \|e^n\|_{\mathcal{D}}^2 \lesssim \Delta t^{2-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2.$$

$$\sup_{n\Delta t \leq T} \|e^n\|_{\mathcal{D}}^2 \lesssim \sup_{n\Delta t \leq T} e^{2n\Delta t^{1+\mu}} \Delta t^{1-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2 \lesssim \Delta t T^3 \|f\|_{H^2(\mathbb{R}^+)}^2.$$

Convergence analysis

Some ingredients:

- Usual consistency of the interior scheme
- Compatibility condition ($a < 0$) : homogeneous Dirichlet/initial data
- Exponential decrease in space of the boundary layer
- Consistency of the several approximations of the trace at the boundary

"GKS estimate" for the consistency error analysis

$$\forall \Delta t \in (0, 1], \forall \gamma > 0, \forall f \in H_0^2(\mathbb{R}^+) :$$

$$\|e\|_{\mathcal{D}, \gamma}^2 \lesssim \left(1 + \frac{1}{\gamma}\right) \Delta t^2 \|f\|_{H^2(\mathbb{R}^+)}^2, \quad \|\eta\|_{\partial, \gamma}^2 \lesssim \Delta t^2 \|f\|_{H^1(\mathbb{R}^+)}^2.$$

Strong stability result for Dirichlet (and zero initial data):

$$\frac{\gamma}{1 + \gamma \Delta t} \|e\|_{\mathcal{D}, \gamma}^2 + \|e\|_{\partial, \gamma}^2 \lesssim \frac{1 + \gamma \Delta t}{\gamma} \|e\|_{\mathcal{D}, \gamma}^2 + \|\eta\|_{\partial, \gamma}^2$$

Let $\gamma = \Delta t^\mu$ with $\mu > 0$

$$\sum_{n \geq 0} \Delta t e^{-2n\Delta t^{1+\mu}} \|e^n\|_{\mathcal{D}}^2 \lesssim \Delta t^{2-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2.$$

$$\sup_{n\Delta t \leq T} \|e^n\|_{\mathcal{D}}^2 \lesssim \sup_{n\Delta t \leq T} e^{2n\Delta t^{1+\mu}} \Delta t^{1-3\mu} \|f\|_{H^2(\mathbb{R}^+)}^2 \lesssim \Delta t T^3 \|f\|_{H^2(\mathbb{R}^+)}^2.$$

Consequence: semigroup estimate for the discrete IBVP

$$\text{for } \sum_{\sigma=0}^k \alpha_{\sigma} u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_{\sigma} \sum_{\ell=-r}^p a_{\ell} u_{j+\ell}^{n+\sigma} = 0, \quad u_j^n = 0 \text{ (boundary)}, \quad u_j^n = f_j^n \text{ (initial)}.$$

Theorem (B. & COULOMBEL, 2017)

Consider *initial data* $f \in H^2(\mathbb{R}^+)$ satisfying the compatibility conditions

$$\begin{cases} f(0) = 0, & \text{if } a < 0, \\ f(0) = f'(0) = 0, & \text{if } a > 0. \end{cases}$$

Suppose the above scheme (with *zero source data* and *zero boundary data*) to be *consistent*, *Cauchy stable*, and *without travelling or glancing wave packets* (H). Then

$$\sup_{n\Delta t \leq T} \sum_{j \geq 0} \Delta x |u_j^n|^2 \leq C \left(\|f\|_{L^2(\mathbb{R}_+)}^2 + \Delta t T^3 \|f\|_{H^2(\mathbb{R}_+)}^2 \right).$$

- $u^{\text{app}} = u^{\text{int}} + u^{\text{bl}} = u + e$
- Boundary layer description $\Rightarrow \|u^{\text{app}}\| \lesssim \|f\|_{L^2(\mathbb{R}_+)}^2$
- $\|u\| \leq \|u^{\text{app}}\| + \|e\|$

First experiments, problematics and strategies

Stability theory for the initial boundary value problem

Dirichlet boundary layer and semigroup estimate for discrete IBVP

High order outflowing extrapolation

- Context and notations

- Numerical experiments

- Improved convergence estimate

Nonhomogeneous problems on bounded domains

Rightgoing transport equation

$$\partial_t u + \partial_x u = 0, \quad x \in (0, 1),$$

$$u(0, x) = f(x),$$

$$u(t, 0) = g(t).$$

Data assumptions

- Smoothness $f \in H^{k+1}(0, 1)$ and $g \in H^{k+1}(0, T)$
- compatibility conditions at order k across the separatrix characteristic line

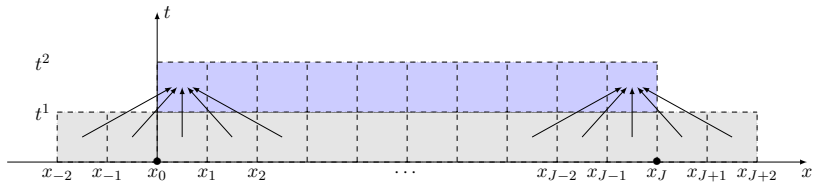
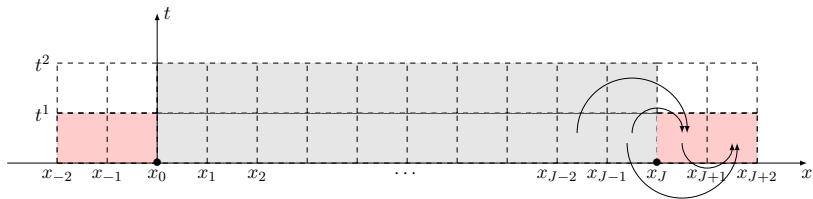
Schemes under consideration

ℓ^2 stable, consistent order k one-step explicit scheme

$$u_j^{n+1} = \sum_{\ell=-r}^p a_\ell u_{j+\ell}^n, \quad 1 \leq j \leq J$$

Discrete boundary conditions

- Neumann extrapolation at order $k_b \geq 0$ at $x = 1$
GOLDBERG '77: strong stability analysis for dissipative or unitary schemes
- Inverse Lax-Wendroff procedure at $x = 0$ (not presented here)
TAN & SHU '10, FILBET & YANG'13, DAKIN'18



Right outflowing boundary condition

Neumann extrapolation at order $k_b \geq 0$

$$(D_-^{k_b} u^n)_{J+\ell} = 0, \quad 1 \leq \ell \leq p \quad \text{where } (D_- v)_j := v_j - v_{j-1}.$$

- $k_b = 0$: homogeneous Dirichlet b.c.
- $k_b = 1$: $u_J = u_{J+1} = \dots = u_{J+p}$
- $k_b = 2$: $u_J - u_{J-1} = u_{J+1} - u_J = u_{J+2} - u_{J+1} = \dots u_{J+p} - u_{J+p-1}$

Lemma (Consistency of extrapolation)

Let $m \in \mathbb{N}$. There exists $C > 0$ such that for any $v \in H^{m+1}(\mathbb{R})$ and $\Delta x > 0$

$$|(D_-^m v)_{J+\ell}| \leq C \Delta x^m \|v^{(m)}\|_{H^1(\mathbb{R})}, \quad \ell = 1, \dots, p.$$

where we set for $j \in \mathbb{Z}$, $v_j := \frac{1}{\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} v(y) dy$.

Previous convergence estimate

COULOMBEL & LAGOUTIÈRE '20 : homogeneous Dirichlet inflow and k_b -Neumann outflow b.c, handling with ℓ^2 -stable explicit two-time level schemes

- no ℓ^∞ stability assumption
- no dissipativity assumption
- proved by energy method

Theorem

There exists a constant $C = C(T) > 0$ such that any integer $J \in \mathbb{N}^*$, any data $f \in H^{k+1}(0, 1)$, the solution (u_j^n) satisfies

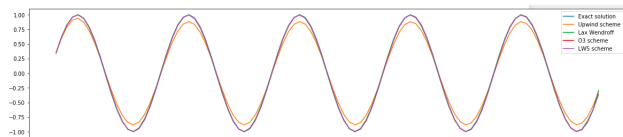
$$\sup_{0 \leq n \Delta t \leq T} \left(\sum_{1 \leq j \leq J} \left| u_j^n - \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(t^n, x) dx \right|^2 \Delta x \right)^{1/2} \leq C \Delta x^{\min(k, k_b)} \|f\|_{H^{k+1}(0,1)},$$

and therefore (crude bound)

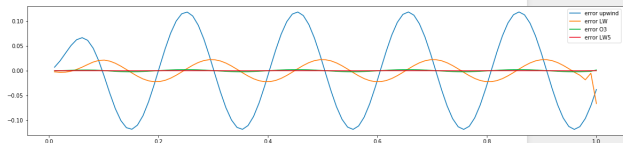
$$\sup_{0 \leq n \Delta t \leq T} \sup_{1 \leq j \leq J} \left| u_j^n - \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(t^n, x) dx \right| \leq C \Delta x^{\min(k, k_b) - 1/2} \|f\|_{H^{k+1}(0,1)}$$

Numerical experiments ($k_b = k$)

Solutions



Local errors

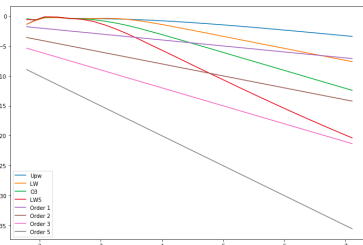


Upwind scheme
Order Linf 0.9988596203602322
Order L2 0.9989909379088449

LW scheme
Order Linf 1.9871218389248466
Order L2 2.019623049148636

O3 scheme
Order Linf 3.022325519501991
Order L2 3.0226590546788894

LW5 scheme
Order Linf 4.714386016709609
Order L2 5.028339253325648



Boundary layer expansions for extrapolation b.c

Boundary equations

$$u^{\text{bl}}(j, t) = \sum_{\ell=1}^p z_{\ell}(t) \rho_{j-J}^{(\ell)}, \quad j \leq J \quad (1)$$

$$(D_-^{k_b} u^{\text{bl}}(\cdot, t))_{J+\ell} + (D_-^{k_b} u^{\text{int}}(\cdot, t))_{J+\ell} = 0, \quad \ell = 1, \dots, p \quad (2)$$

The linear system involves the following $p \times p$ (very) generalized Vandermonde matrix:

$$A_{k_b} = \begin{pmatrix} (D_-^{k_b} \rho^{(1)})_1 & \dots & (D_-^{k_b} \rho^{(p)})_1 \\ \vdots & & \vdots \\ (D_-^{k_b} \rho^{(1)})_p & \dots & (D_-^{k_b} \rho^{(p)})_p \end{pmatrix},$$

and the order k_b consistency of the extrapolation b.c

Lemma

The matrix A_{k_b} is invertible.

Consequence

For $f \in H^{k_b+1}((-\infty, L))$, there exists a unique boundary layer profile. It satisfies:

$$\sup_{n \in \mathbb{N}} \|u^{\text{bl}}\| \lesssim \Delta x^{k_b+1/2} \|f\|_{H^{k_b+1}}$$

Improved convergence result for $k_b < k$

Theorem (Cemracs '19)

There exists a constant $C = C(T) > 0$ such that any integer $J \in \mathbb{N}^*$, any data the solution (u_j^n) satisfies

$$\sup_{0 \leq n \Delta t \leq T} \left(\sum_{1 \leq j \leq J} \left| u_j^n - \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(t^n, x) dx \right|^2 \Delta x \right)^{1/2} \leq C \Delta x^{k_b + 1/2} \|f\|_{H^{k+1}(0,1)},$$

and therefore (crude bound)

$$\sup_{0 \leq n \Delta t \leq T} \sup_{1 \leq j \leq J} \left| u_j^n - \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(t^n, x) dx \right| \leq C \Delta x^{k_b} \|f\|_{H^{k+1}(0,1)}$$

- $u^{\text{app}} = u^{\text{int}} + u^{\text{bl}}$
- $\|u - u^{\text{int}}\| \leq \|u^{\text{bl}}\| + \|u - u^{\text{app}}\|$
- Boundary layer estimate $\Rightarrow \|u^{\text{bl}}\| \lesssim \Delta x^{k_b + 1/2} \|f\|_{H^{k+1}}^2$
- Consistency $\|u - u^{\text{app}}\| \lesssim \Delta x^k \|f\|_{H^{k+1}}^2$

Conclusions

Discrete boundary layer expansions

- improved description of the numerical solutions, overcome consistency limitations
 - for (MOL) **multistep** schemes with Dirichlet boundary conditions, with close to optimal semigroup estimate for the IBVP
 - for **high-order** schemes on one-dimensional bounded domains
- boundary layer structure (high frequency feature) $\neq$ equivalent equation of the scheme (low frequency properties).

On the side of stability analysis:

- ★ Discrete approx. of systems with mixed inflowing/outflowing boundary structures, e.g. hyperbolic relaxation systems (T.H.T Nguyễn and N. Seguin).
- ★ Numer. tools for stability analysis of high order b.c. (P. Le Barbenchon and N. Seguin).

Perspectives:

- ★ more general b.c. and overcome (H) assumption (**other discrete wave packets**)
- ★ Export the tool to multidimensional (simple) situations
- ★ High order numerical coupling through discrete traces

